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Integration by parts formulas and rotationally invariant Sobolev calculus on free loop spaces

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We give formulas for integration by parts over the path space and over the loop space of a manifold. We define Sobolev spaces and an Ornstein–Uhlenbeck operator on the loop space. We find some functionals which belong to all the Sobolev spaces.

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Introduction

There is a link between the equivariant cohomology of the free loop space of a compact oriented simply connected spin manifold and the index theorem, see refs. [3] and [6]. In refs. [11] and [12] Chen forms, by which we mean the differential forms on loop spaces constructed using Chen's iterated integrals [7], were used as a means of understanding and formalising this link. The motivating idea is to use the relation between the cyclic homology of the free loop space of M. In ref. [13] the project of understanding some of the analytical properties of Chen forms by discussing the L^p-theory of Chen forms is started. Our purpose here is to examine the Sobolev calculus on the free loop space. One of our key requirements is that we want a theory which is invariant under the action of the circle on the free loop space given by rotating loops.

In ref. [14] we study two versions of the Sobolev calculus on the free loop space which are both invariant under rotations of loops. More precisely we work on $\mathbb{R}^d$ with metric $\sum g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$, where the g_{ij} are smooth bounded functions. Let Δ be a scalar Laplacian acting on functions defined by this metric; thus $\Delta = \sum g^{ij}\,\partial^2/\partial x_i\,\partial x_j$. Then Δ is a uniformly elliptic second order differential operator. Let $p_t(x, y)$ be the heat kernel of Δ.

Now let Ω_x be the space of loops in $\mathbb{R}^d$ which start at x and return to x after a time period 1. Let $\mathrm{d}P_1^{x,x}$ be the Brownian bridge measure on Ω_x. Let Ω be the free loop space of $\mathbb{R}^d$. Then, following Bismut [6], we introduce the measure $\mu = p_1(x,x)\,\mathrm{d}P_1^{x,x}\,\mathrm{d}x$ on Ω. This measure is the unique measure of the form $f(x)\,\mathrm{d}P_1^{x,x}\,\mathrm{d}x$ on Ω which is invariant under rotations of loops [10].

Let $\mathscr{H}(\mathbb{R}^d)$ be the Hilbert space of paths H: $[0,1]\to\mathbb{R}^d$ which are absolutely continuous with square integrable derivative, equipped with the norm

$$\|H\|^2 = \int_0^1 |H(t)|^2\,\mathrm{d}t + \int_0^1 \left|\frac{\mathrm{d}H(t)}{\mathrm{d}t}\right|^2 \mathrm{d}t\,.$$

Now, as in ref. [13], for each loop $w\in\Omega$ we consider the Hilbert spaces $\mathscr{H}_w$ consisting of the vector fields along the loop w of the form $X=\tau H$ with periodicity assumption where $(\tau H)_t = \tau_t(w)H_t$ with $\tau_t(w)$ being stochastic parallel transport along the loop w. These Hilbert spaces form a measurable field of tangent Hilbert spaces on the loop space and they will play the role of the tangent spaces of the loop space.

The basic tool in setting up this Sobolev calculus is integration by parts formulas. In ref. [14] we use the Peano approximation to the diffusion associated to Δ, which leads to suitable finite dimensional approximations to the Bismut measure μ.

In ref. [13] we suppose that Δ is the Laplace–Beltrami operator on a compact Riemannian manifold. In this case the measure μ has finite mass. We show that integration by parts formulas can be proved using the above choice of tangent spaces to the loop space. To get a more intrinsic version of these formulas the basic tool is a probabilistic representation, due to Bismut [5] (see also ref. [16]) of grad $\log(p_t(x,y))$. In the case of a compact Riemannian manifold the measure μ has finite mass and so the difficulties caused by the fact that $\mathbb{R}^d$ is not compact disappear. This allows us to generalise to our case the definition of connections ∇^t and ∇^{inv} which leads to the same spaces of smooth functions. The solutions of stochastic differential equations are smooth without any assumptions of locality. Also series of iterated integrals whose introduction is motivated by ref. [8]. Moreover, as in ref. [14], we define a Skorohod anticipative integral (see ref. [17] for the flat case) and an Ornstein–Uhlenbeck operator L such that these functions belong to the domain of L^p for each $p>0$. The reader may consult refs. [1,2] for other models.

1. Integration by parts and Sobolev calculus on the path space

Let M be a compact Riemannian manifold of dimension d and let $P(M)$ be the space of all continuous maps w: $[0,1]\to M$. Let ν be the measure $p_1(x,y)\,\mathrm{d}P_1^{x,y}\,\mathrm{d}x\,\mathrm{d}y$ on $P(M)$, where $p_1(x,y)$ is the heat kernel of M and $\mathrm{d}P_1^{x,y}$

is the Brownian bridge measure on $P_{x,y}(M)$, the space of paths joining x to y. We use the notation E_P for expectations computed with respect to this measure.

Following ref. [13] we take for the tangent space at a path $w \in P(M)$ the Hilbert space $\mathcal{H}_w$ of vector fields along w of the form $X_t = \tau_T H_t$, where H_t is a path in $T_{w_0}M$, and τ_t is stochastic parallel transport along w. This space is equipped with the norm

$$\|X_t\|^2 = |H_0|^2 + \int_0^1 \left| \frac{dH_s}{ds} \right|^2 ds . \tag{1.1}$$

Since we are working on the path space there is no circle action and we do not need to choose an invariant norm.

To define a connection ∇ on the tangent spaces to $P(M)$ suppose X and Y are vector fields on $P(M)$, that is, sections of the field of Hilbert spaces $\mathcal{H}$. Now $Y = \tau K$ and we can write out K in components

$$K(w) = \sum_{i=1}^{d} k^i(w) V_i(w_0) ,$$

where the k^i are functions on $P(M)$ and the V_i are vector fields on M. Now define $\nabla_X K$ by the formula

$$(\nabla_X K)_w = \sum_{i=1}^{d} \langle dk^i, X \rangle_w V_i(w_0) + k^i(w) (\nabla_{X_0} V_i)_{w_0} , \tag{1.2}$$

and define $\nabla_X Y$ by $\nabla_X Y = \tau \nabla_X K$.

If F is a function on $P(M)$ then $\nabla^r F$ is defined as usual in Riemannian geometry and is an r-tensor. That is, for each $w \in P(M)$, $\nabla^r F_w : \mathcal{H}_w \times \cdots \times \mathcal{H}_w \to \mathbb{R}$ is a multilinear function. To prove that this operator is closable, we work with the basis for $\mathcal{H}_w$ given in ref. [14 (III)] and to get the necessary estimates we need an integration by parts formula.

Let X_0 be a smooth vector field on M and let $h = \sum h^i V_i$, where the components $h^i : [0, 1] \to \mathbb{R}$ are functions and the V_i are vector fields on M. Now let $X(w)$ be the vector field along w defined by $X_t(w) = \tau_t(X_0(w_0) + H_t(w))$, where $H_t = \int_0^t h_s \, ds$.

Theorem 1.1. *For any smooth function $f: M^k \to \mathbb{R}$ and any set of times $t = (t_1, ..., t_k)$ with $0 < t_1 < \cdots < t_k < 1$ let F be the function on $P(M)$ given by $F(w) = f(w(t_1), ..., w(t_k))$. Then we have the following integration by parts formula:*

$$E_P[\langle dF, X \rangle] = E_P\left[F\left(\operatorname{div} X_0(w_0) + \int_0^1 \langle \tau_s h_s, \delta w_s \rangle + \frac{1}{2} \int_0^1 \langle S_{\tau_s H_s}, \delta w_s \rangle \right) \right], \tag{1.3}$$

where X is a vector field on $P(M)$, S is the Ricci tensor of M and δ the Itô integral.

We will discuss the proof of this theorem later (see ref. [4] for the scheme in the flat case). Our present purpose is to develop a Sobolev calculus on the path space $P(M)$. As usual we define the Sobolev (p, q)-norm of a function F on the path space by

$$\|F\|_{P,p,q} = \sum_{r\leqslant p} E_P[\|\nabla^r F\|^q]^{1/q}, \tag{1.4}$$

and the associated Sobolev space $W_{P,p,q}$. It is clear that for $p'\geqslant p$ and $q'\geqslant q$,

$$\|\cdot\|_{P,p,q} \leqslant \|\cdot\|_{P,p',q'} \quad \text{and} \quad W_{P,p,q} \supset W_{P,p',q'} .$$

Now define the space of smooth functions $W_{p,\infty} = \bigcap W_{P,p,q}$. It follows that $W_{P,\infty}$ is an algebra under pointwise multiplication of functions.

Next we prove two theorems which show that natural functions on $P(M)$ are smooth in the above sense.

Theorem 1.2. *Let Z_t be the solution of the Stratonovitch differential equation in $\mathbb{R}^d$*

$$\mathrm{d}Z_t = B(Z_t, w_t, w_0)\,\mathrm{d}w_t + C(Z_t, w_t, w_0)\,\mathrm{d}t\,, \quad Z_0 = y\,, \tag{1.5}$$

where B and C have bounded derivatives of order $\geqslant 1$. Then Z_t is in $W_{P,\infty}$.

In particular this theorem shows that parallel transport τ_t and a function $f(w_t)$ where f is smooth are both in $W_{P,\infty}$. Our second theorem is motivated by ref. [8]; see ref. [13] for the L^p version.

Theorem 1.3. *Let f^n be smooth functions and w_i^n one-forms on M. Let F be the function*

$$F(w) = \sum_n f^n(w_0) \int_{0\leqslant s_1\leqslant\cdots\leqslant s_n\leqslant 1} w_1^n(\mathrm{d}w_{s_1})\cdots w_n^n(\mathrm{d}w_{s_n})\,. \tag{1.6}$$

Then F is in $W_{P,\infty}$ if the power series

$$\psi_k(z) = \sum \|f^n\|_{k,\infty}\left(\frac{\prod_{i=1}^{\infty}\|w_i^n\|_{k,\infty}}{\sqrt{n!}}\right)z^n \tag{1.7}$$

has infinite radius of convergence for any k. Here $\|w_i^n\|_{k,\infty}$ is the uniform C^k norm of w_i^n.

Now let us discuss the proof of the integration by parts formula, theorem 1.1. First recall an integration by parts formula due to Bismut [5]. Let $P_x(M)$ be the space of continuous paths in M which start at x. Given $w \in P_x(M)$, let Y_t be the vector field along w_t which is the solution of the stochastic differential equation

$$DY_t = (-\tfrac{1}{2}SY + \tilde{h})\,\mathrm{d}t\,, \quad Y_0 = 0\,, \tag{1.8}$$

where D is the covariant derivative. Here the function $\tilde{h}$ is previsible and bounded in L^2. As above let f be a smooth function on M^n where there are n factors of M and fix a sequence of times $t = (t_1, \ldots, t_n)$. This determines a function $F: P_x(M) \to \mathbb{R}$, $w \mapsto f(w_{t_1}, \ldots, w_{t_n})$. From ref. [8] we get the following formula:

$$E_x[\langle \mathrm{d}F, Y\rangle] = E_x\left[F\int_0^1 \langle \tilde{h}_s, \delta w_s\rangle\right], \tag{1.9}$$

where $\int_0^1 \langle \tilde{h}_s, \delta w_s\rangle$ is the Itô integral.

By taking $X_t = \tau_t \int_0^t h_s\,\mathrm{d}s$, $\tilde{h} = \tau_s h_s + \frac{1}{2}S_{X_s}$ we deduce the following formula due to Driver [9]:

$$E_x[\langle \mathrm{d}F, Y\rangle] = E_x\left[F\left(\int_0^1 \langle \tau_s h_s, \delta w_s\rangle + \int_0^1 \langle \tfrac{1}{2}S_{X_s}, \delta w_s\rangle\right)\right]. \tag{1.10}$$

To prove theorem 1.1 we follow the method of ref. [14] but use a different finite approximation to νp. Choose a cylinder set C_s, where $s = (s_1, \ldots, s_n)$ is a sequence of times which includes $(t_1, \ldots, t_k)$. Now restricting to C_s we have

$$\int_{C_s} F = \int_{M^{N+2}} p_{s_1}(x, x_1) p_{s_2 - s_1}(x_1, x_2) \cdots p_{1 - s_N}(x_N, y)$$
$$\times f(x_{j_1}, \ldots, x_{j_k})\,\mathrm{d}x\,\mathrm{d}x_1 \cdots \mathrm{d}x_N\,\mathrm{d}y\,,$$

where j_r is determined by $t_r = s_{j_r}$. Approximate the path w by using geodesics joining w_{s_i} to w_{s_i+1}. Only the terms with w_{s_i} and $w_{s_{i+1}}$ close to each other contribute to the limit as $N \to \infty$. By using the parallel transport operator along this broken geodesic we get an approximation X_t^N of the vector field X_t in the case where h_t is piecewise smooth,

$$X_{t_{i+1}}^N = \tau_{t_{i+1}, t_i}^N (X_{t_i}^N + (t_{i+1} - t_i)\tau_{t_i}^N h_{t_i})\ .$$

Here τ_{t_{i+1}, t_i}^N is parallel transport from w_{t_i} to $w_{t_{i+1}}$ along the geodesic joining these two points. Now we pass to the limit $N \to \infty$.

Lemma 1.4. *Let S and T be two stopping times. There exist semi-martingale processes A, B, C and D such that A_s, B_s, C_s and D_s are smooth functions of w_s with values in a suitable space of forms such that*

$$\sum_{S < t_i < T} \nabla_{X_{t_i}^N} \log p_{t_{i+1} - t_i}(w_{t_i}, w_{t_{i+1}}) + \nabla_{X_{t_{i+1}}^N} \log p_{t_{i+1} - t_i}(w_{t_i}, w_{t_{i+1}}) \tag{1.11}$$

converges as $N \to \infty$ to

$$\int_S^T (\langle A_s(X_s), \delta w_s\rangle + \langle B_s(\tau_s h_s), \delta w_s\rangle + C_s(X_s)\,\mathrm{d}s + D_s(\tau_s h_s)\,\mathrm{d}s)\,. \quad (1.12)$$

Proof. Following ref. [5, theorem 2.14] we have

$$\begin{aligned}&\nabla_{X^N_{t_i}} \log p_{t_{i+1}-t_i}(w_{t_i}, w_{t_{i+1}})\\ &= E^{t_{i+1}-t_i}_{w_{t_i}, w_{t_{i+1}}}\left[\frac{1}{t_{i+1}-t_i}\int_{t_i}^{t_{i+1}} \langle \tilde{\tau}_s X^N_{t_i}, \mathrm{d}w_s\rangle\right],\end{aligned} \quad (1.13)$$

where we take the Stratonovitch integral along the Brownian path w_s joining w_{t_i} to $w_{t_{i+1}}$ in a time $t_{i+1}-t_i$. Here $\tilde{\tau}_s$ is the solution of the equation $D\tilde{\tau}_s = -\frac{1}{2}S\tilde{\tau}_s\,\mathrm{d}s$, $\tilde{\tau}_0 = 1$.

By reversing time along the path w_s we get

$$\begin{aligned}\nabla_{X^N_{t_{i+1}}} \log p_{t_{i+1}-t_i}(w_{t_i}, w_{t_{i+1}}) &= -E^{t_{i+1}-t_i}_{w_{t_i}, w_{t_{i+1}}}\left[\frac{1}{t_{i+1}-t_i}\right.\\ &\left.\times \int_{t_i}^{t_{i+1}} \langle \bar{\tau}_s \bar{\tau}^{-1}_{t_{i+1}-t_i} \tau^N_{t_{i+1},t_i}(X^N_{t_i} + (t_{i+1}-t_i)\tau^N_{t_i} h_{t_i}), \mathrm{d}w_s\rangle\right],\end{aligned} \quad (1.14)$$

where $D\bar{\tau}_s = \frac{1}{2}S\bar{\tau}_s\,\mathrm{d}s$. Thus we need to study the convergence of

$$\begin{aligned}\sum_{S<t_i<T} \frac{1}{t_{i+1}-t_i} E^{t_{i+1}-t_i}_{w_{t_i}, w_{t_{i+1}}}&\left[\int_{t_i}^{t_{i+1}} \langle \tilde{\tau}_s X^N_{t_i} - \bar{\tau}_s \bar{\tau}^{-1}_{t_{i+1}-t_i} \tau^N_{t_{i+1}-t_i}\right.\\ &\left.\times (X^N_{t_i} + (t_{i+1}-t_i)\tau^N_{t_i} h_{t_i}), \mathrm{d}w_s\rangle\right].\end{aligned}$$

Since $\tilde{\tau}_s$ and $\bar{\tau}_s$ differ from τ_s by a term in $t_{i+1}-t_i$, the most difficult part is to study the behaviour of

$$\begin{aligned}K^N(S,T) &= \sum_{S<t_i<T} \frac{1}{t_{i+1}-t_i}\\ &\times E^{t_{i+1}-t_i}_{w_{t_i}, w_{t_{i+1}}}\left[\int_{t_i}^{t_{i+1}} \langle \tau_s X^N_{t_i} - \tau_s \tau^{-1}_{t_{i+1}-t_i} \tau^N_{t_i,t_{i+1}} X^N_{t_i}, \mathrm{d}w_s\rangle\right].\end{aligned} \quad (1.15)$$

The type of cancellation which occurs in ref. [14] will also occur here. To see this we study the measure

$$\mu_{t,x}(f)=E_x\left[\frac{1}{t}\int_0^t \langle \tau_s-\tau_s\tau_t^{-1}\tau_t^N, \mathrm{d}w_s\rangle f(w_t)\right] \tag{1.16}$$

for Brownian motion starting at x. We can suppose that over $[S, T]$ the process lies in $\mathbb{R}^d$. If x and y are close to each other the density of this measure has an expansion in the form

$$q_t(x,y)=\frac{\exp(-d(x,y)^2/2t)}{t(2\pi t)^{d/2}}\sum a_i(x,y)t^{i/2}, \tag{1.17}$$

where the $a_i(x,y)$ are smooth. Here $d(x,y)$ is the Riemannian distance function. Now we use the fact that $E[\int_0^1 g_s\,\mathrm{d}w_s]$ for a flat Brownian loop bridge starting at 0 and ending at 0 after time 1. By working in normal coordinates (see refs. [15,18] for similar computations) we deduce that $a_0(x,y)=a_1(x,y)=0$ provided x and y are close enough. Moreover $a_2(x,x)=a_3(x,x)=0$; to prove that $a_3(x,x)=0$ we need to use the fact that the expectation over a flat Brownian bridge of iterated integrals of the form

$$\int_{0<s_1<s_2<s_3<1} \mathrm{d}w_{s_1}^{i_1}\,\mathrm{d}w_{s_2}^{i_2}\,\mathrm{d}w_{s_3}^{i_3}$$

is zero. Now it follows that

$$\begin{aligned} a_2(x,y)&=a_2^{(1)}(x,x)(y-x)+a_2^{(2)}(x,x)(y-x)^2+\mathrm{o}(|x-y|^3),\\ a_3(x,y)&=a_3^{(1)}(x,x)(y-x)+a_3^{(2)}(x,x)(y-x)^2+\mathrm{o}(|x-y|^3),\\ a_4(x,y)&=a_4(x,x)+\mathrm{o}(|y-x|). \end{aligned} \tag{1.18}$$

The heat kernel has an expansion in the form

$$p_t(x,y)=\frac{\exp(-d(x,y)^2/2t)}{(2\pi t)^{d/2}}\sum b_i(x,y)t^{i/2}$$

and we deduce that if x_t lies in a subset of $\mathbb{R}^d$ over the interval $[S, T]$ then

$$\begin{aligned} K^N(S,T)=&\sum_{S<t_i<T}\langle A'(w_{t_i})X_{t_i}^N, w_{t_{i+1}}-w_{t_i}\rangle\\ &+\sum_{S<t_i<T}(B'(w_{t_i})X_{t_i}^N)(t_{i+1}-t_i)\\ &+\sum_{S<t_i<T}\langle C'(w_{t_i})X_{t_i}^N, w_{t_{i+1}}-w_{t_i}, w_{t_{i+1}}-w_{t_i}\rangle\\ &+\sum_{S<t_i<T}\langle D'(w_{t_i})X_{t_i}^N, \mathrm{o}(|w_{t_{i+1}}-w_{t_i}|^3)\rangle. \end{aligned} \tag{1.19}$$

We can now pass to the limit. This limit has an intrinsic character, that is, it does not depend on the choice of local coordinates, since K^N does, and our theorem follows by converting the Itô integrals over $\mathbb{R}^d$ which occur in the above expression for $K^N(S, T)$ into Itô integrals over M, which have an intrinsic meaning by considering the Itô–Meyer decomposition of the process $K(S, \min(t, T))$. □

Lemma 1.5. *Let S and T be two stopping times. There exist semi-martingale processes A and B such that A_s and B_s are smooth functions of w_s with values in forms such that $\sum_{S<t_i<T} \operatorname{div} X_{t_i}^N$ converges as $N\to\infty$ to*

$$\int_S^T \langle A_s(X_s), \delta w_s\rangle + \int_S^T B_s(X_s)\, \mathrm{d}s\,.$$

Proof. We can suppose that over the interval $[S, T]$ the process lies in a local chart for M. We work in normal coordinates. If τ_s^N is parallel transport along the geodesic joining $w_{t_{i-1}}$ to w_{t_i} for $i\geqslant 1$ it follows that

$$\mathrm{d}\tau_s^N = -\Gamma_{\tau_s^N}(w_{t_{i-1}} + s(w_{t_i} - w_{t_{i-1}}))(w_{t_i} - w_{t_{i-1}})\, \mathrm{d}s\,, \quad \tau_0^N = \tau_{i-1}^N\,, \tag{1.20}$$

where the Christoffel matrix Γ is zero at $w_{t_{i-1}}$, is the analogue of the cancellation which appears in ref. [14] and we deduce that

$$\begin{aligned}\operatorname{div} X_{t_i}^N = {} & A'(w_{t_{i-1}}, X_{t_{i-1}}^N + (t_{i+1} - t_i)\tau_{i-1}^N h_{t_{i-1}})(w_{t_i} - w_{t_{i-1}}) \\ & + B'(w_{t_{i-1}}, X_{t_{i-1}}^N + (t_{i+1} - t_i)\tau_{i-1}^N h_{t_{i-1}})(w_{t_i} - w_{t_{i-1}})^2 \\ & + \mathrm{o}((w_{t_i} - w_{t_{i-1}})^3) + \mathrm{o}(t_i - t_{i-1})\,,\end{aligned} \tag{1.21}$$

and the proof of lemma 1.5 is completed in the same way as the proof of lemma 1.4.

From both previous lemmas we see that

$$\begin{aligned}E_P[\langle \mathrm{d}F, X\rangle] = E_P\Big[F\Big(& \operatorname{div} X_0 + \int_0^1 \langle A(X_s), \delta w_s\rangle \\ & + \int_0^1 \langle B(\tau_s h_s), \delta w_s\rangle + \int_0^1 C(X_s)\, \mathrm{d}s + \int_0^1 D(\tau_s h_s)\Big)\Big].\end{aligned} \tag{1.22}$$

If $X_0 = 0$ we know that the sum of the above integrals is equal to

$$\int_0^1 \langle \tau_s h_s, \delta w_s\rangle + \frac{1}{2}\int_0^1 \langle S_{\tau_s H_s}, \delta w_s\rangle\,.$$

The formula for $E_P[\langle \mathrm{d}F, X\rangle]$ can be extended to all previsible h_s bounded in L^2. Moreover A, B, C, D have the same shape as in ref. [14]. However, if $\bar{A}$, $\bar{B}$, $\bar{C}$, and $\bar{D}$ are processes of this particularly nice type such that for all $X_s = \tau_s H_s$, where $H_s = \int_0^s h_u \, du$ with h previsible and bounded, we have

$$\int_0^1 \langle \bar{A}(X_s), \delta w_s \rangle + \int_0^1 \langle \bar{B}(\tau_s h_s), \delta w_s \rangle + \int_0^1 \bar{C}(X_s) \, ds + \int_0^1 \bar{D}(\tau_s h_s) \, ds = 0 ,$$

then it follows that $\bar{A} = \bar{B} = \bar{C} = \bar{D} = 0$.

If $\bar{B}$ is not zero we can choose a stopping time $T < 1$, with probability strictly positive, and a deterministic process $\tilde{h}_s$ with $\tilde{h}^{(k)}(0) \neq 0$ such that on the condition that $F_T < 1$, if we take $h_s = h_{T+s}$, the sum of the four integrals in the above formula has a density such that from Malliavin calculus we deduce from the same type of argument that $\bar{A} = 0$. For all X_s we have $\int_0^1 \bar{C}(X_s) \, ds + \int_0^1 \bar{D}(\tau_s h_s) \, ds = 0$. But $\bar{D}$ is a smooth function in w_s. We deduce that $\bar{D}(\tau_s)$ has bounded variation, and from stochastic calculus it is constant.

That implies $\bar{C} = 0$. □

2. The Sobolev calculus on the free loop space and anticipative integrals

In order to get an integration by parts formula over the free loop space for periodic vector fields

$$X_t = \tau_t \left[X_0(w_0) + t(\tau_1^{-1}) X_0(w_0) + \int_0^t h_s \, ds \right]$$

with h_s as in the previous section, such that $\int_0^1 h_s \, ds = 0$, we proceed in the same way. First we extend the vector fields over the path space by setting

$$X_t = \varphi(w_0, w_1) \tau_t \left[X_0(w_0) - t X_0(w_0) + t \tau_1^{-1} \tau(w_1, w_0) X_0(w_0) + \int_0^t h_s \, ds \right],$$

where $\varphi = 1$ when $w_0 = w_1$ and has a small compact support around the diagonal such that w_0 and w_1 are joined by a unique geodesic when $\varphi \neq 0$, and $\tau(w_1, w_0)$ is parallel transport along that geodesic. If we study the divergence of the extended vector fields X_t over the path space, the derivative of τ^{N-1} and $\tau^N(w_0, w_1)$ will appear in its approximation $\tilde{k}^N$; but it differs from the polygonal approximation of integration by parts over the loop space by a term in which the derivative of $\tau^N(w_0, w_1)$ appears. So we only have to study the limiting behaviour of the expression $\tilde{k}^N$ for the law $P_{x,x}$. As in ref. [14], we use Malliavin calculus by studying the measure $f \to E_x[\tilde{k}^N f(w_1(x))]$. Since $\tilde{k}^N$ converges to $\tilde{k}$ in the Malliavin sense, we can introduce the function over the space

$$\tilde{F}(w_0): Y(w_0) \to \nabla_{tY(w_0)} \tau_1^{-1} \tau(w_0, w_1) \; .$$

In the Malliavin sense, if $X_t = \tau_t H_t$,

$$\tilde{K}^N \to \operatorname{div} X_0 + \int_0^1 \langle \tau_s \dot{H}_s, \delta w_s \rangle + \frac{1}{2} \int_0^1 \langle S_{X_s}, \delta w_s \rangle - \operatorname{tr} \tilde{F}(w) X_0(w_0) = \tilde{K} \; . \tag{2.1}$$

But over the loop space, div X differs only from $\tilde{K}$ in the derivative of $\tau(w_0, w_1)$. So we take F_w over the loop space:

$$F_w : Y(w_0) \to \nabla_{tY(w_0)} \tau^{-1} \, ,$$

and

$$\operatorname{div} X = \operatorname{div} X_0 + \int_0^1 \langle \tau_s \dot{H}_s, \delta w_s \rangle + \frac{1}{2} \int_0^1 \langle S_{X_s}, \delta w_s \rangle - \operatorname{tr} F(w) X_0(w_0) \; . \tag{2.2}$$

Now we can introduce a connection ∇^0. We first take the subbundle of the tangent space of the loop space of vectors of the type $\tau_t \int_0^t h_s \, \mathrm{d}s$, $\int_0^1 h_s \, \mathrm{d}s = 0$, and the connection ∇^0 over that subspace is the restriction of the previous connection to that subspace. If X_0 is given, there is only one vector field $\tilde{X}(X_0)$ orthogonal to the previous space for the Hilbert structure $\int_0^1 |X_t|^2 \, \mathrm{d}t + \int_0^1 |\dot{H}_t|^2 \, \mathrm{d}t$ (see ref. [13] for that Hilbert structure). $\nabla^0 \tilde{X}(X_0)$ is by definition equal to $\tilde{X}(\nabla X_0)$. We can rotate the connection ∇^0 and we find a connection ∇^t; by averaging we get a connection $\nabla^{\mathrm{inv}} = \int_0^1 \nabla^t \, \mathrm{d}t$ which is invariant under rotation. The Sobolev space for these connections (integration by parts for the orthogonal basis of the Hilbert space used in ref. [14] allows us to define them) satisfies (see ref. [14])

$$C\|\cdot\|_{p,q} \leqslant \|\cdot\|^t_{p,q'} \leqslant \|\cdot\|_{p,q''} \quad \text{for } q < q' < q'' \; . \tag{2.3}$$

Moreover, we have:

Theorem 2.1. *A smooth functional over the path space restricts to a smooth functional over the loop space.*

We deduce from this theorem that theorems 1.2 and 1.3 are still true. Moreover, the zero order part of the Witten current which to F associates

$$\mu\left[\operatorname{tr}_s \tau_1^{-1} \exp\left(-\frac{1}{8}\int_0^1 K(w_s)\,\mathrm{d}s\right) F\right]$$

is a smooth measure in this sense (K is the scalar curvature and $\operatorname{tr}_s \tau_1^{-1}$ the supertrace over the spinor for a spin manifold [12,11]).

In order to handle the Laplacian we can repeat the argument of ref. [14]. If F is a smooth functional over the path space, we will say it is strongly smooth if the kernels $k_I(s_1, ..., s_n)$ of its derivatives satisfy outside the the diagonals the Kolmogorov criterion (see ref. [14] for more details). A vector field $\tau_t H_t$ over the path space is said to be strongly smooth if the kernels $k_I(t, s_1, ..., s_n)$ of its covariant derivatives satisfy outside the diagonals the Kolmogorov criterion, t included. If w_0 and w_1 are close enough, we define the subbundle of the tangent space of vectors such that $X_1 = \tau(w_1, w_0)X_0$ with the Hilbert structure $\int_0^1 |X_t|^2\,\mathrm{d}t + \int_0^1 |\dot{H}_t|^2\,\mathrm{d}t$. $\mathrm{d}F$ being a continuous form, we can define its dual X_F over that Hilbert subbundle and by adding $\varphi(w_0, w_1)$ before X_F, we get $\mathrm{grad}_{\mathrm{ext}}F = \varphi(w_0, w_1)X_F$, which restricts to grad F over the loop space for smooth functionals over the path space.

Theorem 2.2. *If F is strongly smooth,* $\mathrm{grad}_{\mathrm{ext}}F$ *is strongly smooth.*

Moreover, let X be a strong smooth vector field such that $X_1 = \tau(w_1, w_0)X_0$ and $X_t = 0$ if $\varphi(w_1, w_0) = 0$. We consider $\varphi(w_1, w_0)X$ and split it in two parts. The first part is given by $\tau_t \int_0 h_s\,\mathrm{d}s = \tilde{X}$ with $\int_0^t h_s\,\mathrm{d}s = 0$. Since $\tilde{X}_1 = \tilde{X}_0 = 0$, its divergence, if it exists over the path space and is smooth, restricts to the divergence over the loop space. The second part $\bar{X}$ is given by

$$\bar{X}_t = \tau_t[X_0 - tX_0 + t\tau_1^{-1}\tau(w_1, w_0)X_0]\ .$$

Taking the formula given at the beginning for the divergence over the loop space [we do not derive $\tau(w_1, w_0)$], we get $\mathrm{div}_{\mathrm{ext}}\bar{X}$ and we put $\mathrm{div}_{\mathrm{ext}}x = \mathrm{div}_{\mathrm{ext}}\bar{X} + \mathrm{div}\ \tilde{X}$. It is not the divergence over the path space because in the last case we have to derive $\tau(w_1, w_0)$ too. As in ref. [14], we have:

Theorem 2.3. *If X is strongly smooth,* $\mathrm{div}_{\mathrm{ext}}X$ *is strongly smooth.*

This theorem implies that, if F is a strongly smooth functional over the path space, we can restrict F over the loop space and apply all the power of the Ornstein–Uhlenbeck operator $L = \mathrm{div\ grad}$ over the loop space. In particular, we can apply, following ref. [14], all the power of L over the loop space of the functionals defined by theorems 1.1 and 1.2 and get quantities which belong to all of $L^p(\mu)$.

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